7.1 <u>The Standard Normal Curve</u>

The "bell-shaped" curve, or <u>normal curve</u>, is a probability distribution that describes many real-life situations.

<u>Basic Properties</u>

1. The total area under the curve is________.

2. The curve extends infinitely in both directions along the horizontal z-axis.

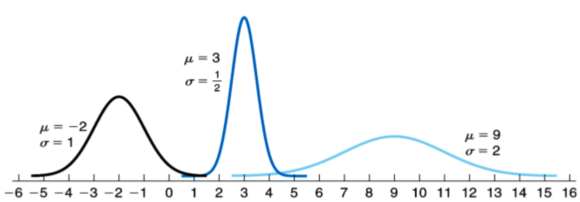
3. The curve is symmetric about__________.

4. Most of the area under the curve lies between _____________and__________.

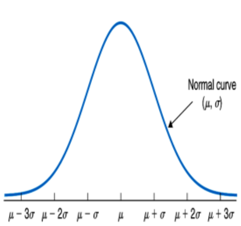
Two items of importance relative to normal distributions are as follows:

- If a variable of a population is normally distributed and is the only variable under consideration, then it has become common statistical practice to say that the **population is normally distributed** or that we have a **normally distributed population**.
- In practice it is unusual for a distribution to have exactly the shape of a normal curve. If a variable's distribution is shaped roughly like a normal curve, then we say that the variable is **approximately normally distributed** or has **approximately a normal distribution**.

Three normal distributions

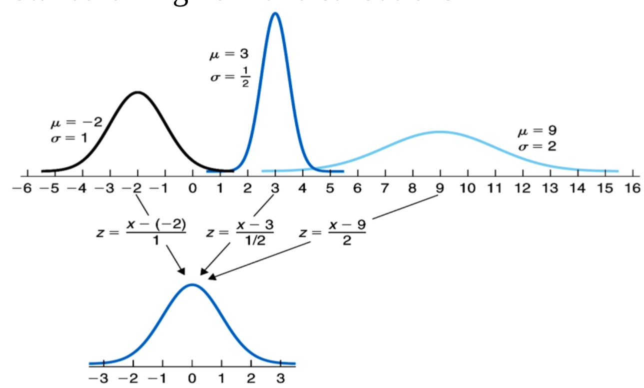


Graph of generic normal distribution.



To find areas under a normal curve, we must first ***standardize*** the distribution, turning the data values x into standardized z-scores:

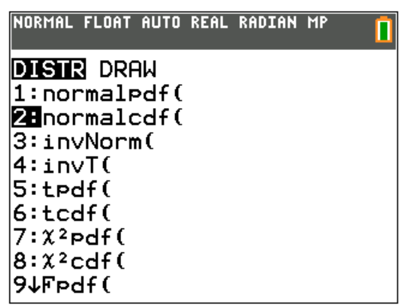
Standardizing normal distributions



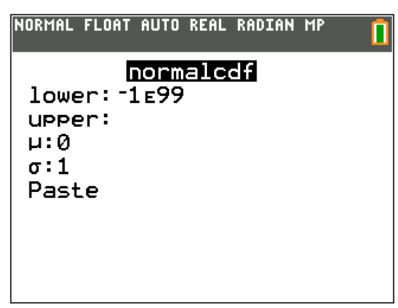
<u>Key Facts:</u>

- If you were to add 10 to every observation in a data set, the mean of the data set would increase by 10 and the standard deviation would remain the same; this is like "shifting" the graph 10 units to the right on the x-axis while keeping the shape intact.
- If you were to combine two distributions together, you would not be allowed to add their means and standard deviations together.

To find areas under the curve, use the **normalcdf** function found in the DIST menu.



To access this menu, press **2nd** and **dist** (above the **vars** button). You will likely never use the normal**pdf** option (option 1).

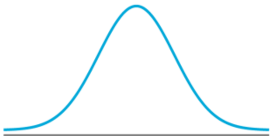


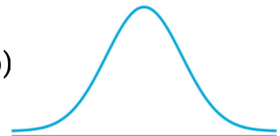
Here is the TI-84 Plus screen. Note that the default values of $\mu = 0$ and $\sigma = 1$ are already listed there.

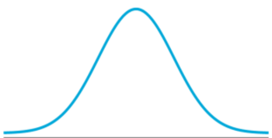
The $-1E99$ is the calculator's way of denoting negative infinity.

For those of you using an older calculator, here is the format to enter:

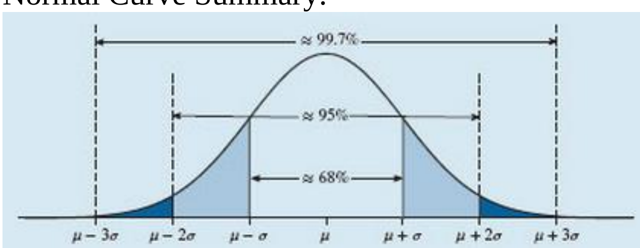
Normalcdf (lowerbound, upperbound [, mu, sigma])
example: Find the shaded area under the standard normal curves:

a) 

b) 

c) 

Normal Curve Summary:



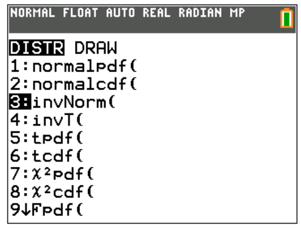
Comparison with Chebyshev's Theorem:

# std. dev.	at least...% Chebyshev	Normal Distribution
0	0	0
1	0	68.26%
2	75%	95.44%
3	89%	99.74%

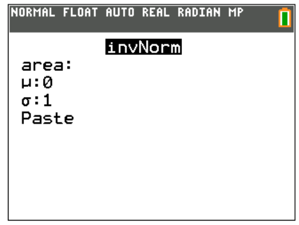
The actual formula for the normal curve is

$y = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$. Technically, the curve changes concavity 1 standard deviation to either side of the mean.

Going Backwards: Given a shaded area, find its corresponding z-value. Use the InvNorm command in the DIST menu:



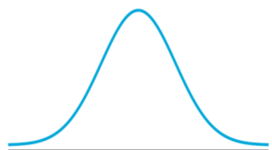
To access this menu, press **2nd** and **dist** (above the **vars** button).

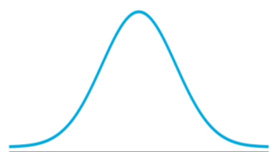


The area is always to the **left** of **z**.

For older calculators:
InvNorm(area [, mu, sigma])

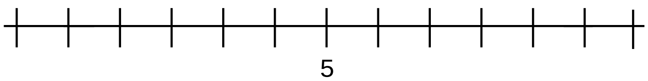
example: Find the associated z-value:

a) 

b) 

The **standard** normal curve had $\mu = 0$ and $\sigma = 1$. However a normal curve refers to a **whole family of curves** defined by μ and σ.

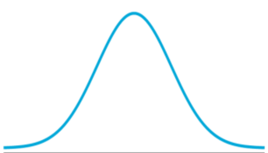
example: Sketch the normal curve with $\mu = 5$ and $\sigma = 2$. (Recall that most of the data lies within $\mu \pm 3\sigma$). On top of your sketch, draw the normal curve with $\mu = 5$ and $\sigma = 0.5$.



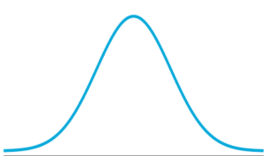
To find areas under **any** normal curve (not just the ones with $\mu = 0$ and $\sigma = 1$) use the **normalcdf** function, entering in the values of mu and sigma:

Normalcdf (lowerbound, upperbound [, mu, sigma])

example: For $\mu = 5$ and $\sigma = 2$, find the area to the right of x=7.5.



example: Find the area between x=3 and x=11.5 when $\mu = 11$ and $\sigma = 4$.

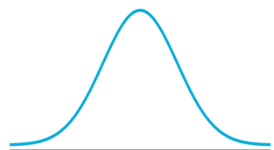


example: Find the area between x=5 and x=11 when $\mu = 4$ and $\sigma = 1.3$.

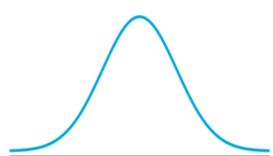
To find particular *x* values, given the area under the normal curve, use the InvNorm command followed by mu and sigma:

InvNorm (area [, mu, sigma])

example: $\mu = 150$ and $\sigma = 20$. Find the *x*-value with an area of 0.1056 to its left.



example: Assume that the mean length of an adult cat's tail is 13.5 inches with a standard deviation of 1.5 inches. Complete the following sentence: 13% of adult cats have tails that are longer than________________inches.



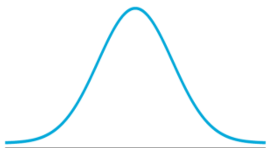
7.2 Applications of the Normal Distribution

A population is said to be normally distributed if percentages of the population are approximately equal to areas under the normal curve.

*heights, ages, test scores, IQ's

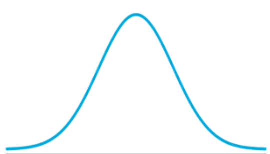
Example: Assume the heights of US males over 18 years old are approximately normally distributed with $\mu = 68$" and $\sigma = 3$". (I made these numbers up!)

Find the percentage of US men between 6' and 6'4" tall.

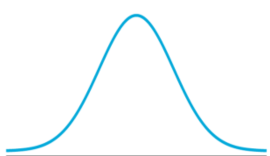


Example: The mean travel time to work in New York State is 29 minutes. Let x be the time, in minutes, that it takes a randomly selected New Yorker to get to work on a randomly selected day. If the travel times are normally distributed with a standard deviation of 9.3 minutes, find...

a) $P(\mathbf{x} < 45)$



b) $P(20 \le \mathbf{x} \le 30)$



c) Interpret your results to parts (a) and (b).

Example: The weights of a certain type of adult bird are approximately normally distributed with a mean of 1384 grams and a standard deviation of 159 grams.

a. What proportion weigh between 1100 and 1200 grams?

b. What is the probability that a randomly selected bird will weigh more than 1500 grams?

c. Is it unusual for an adult bird of this type to weigh more than 1550 grams?

Example: The average charitable contribution itemized per income tax return is \$758. Suppose the distribution of contributions is normal with a standard deviation of \$102. Find the limits for the middle50% of contributions.

Example: To qualify to become a security guard at a certain firm, applicants need to be tested for stress tolerance. The scores are normally distributed with a mean of 61 and a standard deviation of 8. If only the top 15% are selected, find the cutoff score.

7.3 <u>Central Limit Theorem</u>

A **<u>sampling error</u>** is the error resulting from using a ***sample*** instead of a ***census*** to estimate a population quantity. The larger the sample size, the smaller the sampling error in estimating a population mean μ by a sample mean $\bar{x}$.

An illustration:

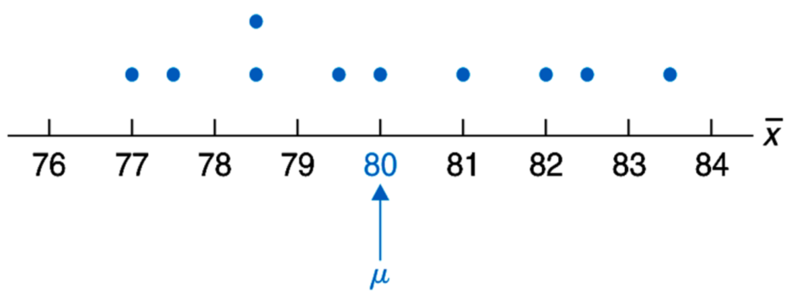
Heights of the five starting players

Player	A	B	C	D	E
Height	76	78	79	81	86

Possible samples and sample means for samples of size two

Sample	Heights	$\bar{x}$
A, B	76, 78	77.0
A, C	76, 79	77.5
A, D	76, 81	78.5
A, E	76, 86	81.0
B, C	78, 79	78.5
B, D	78, 81	79.5
B, E	78, 86	82.0
C, D	79, 81	80.0
C, E	79, 86	82.5
D, E	81, 86	83.5

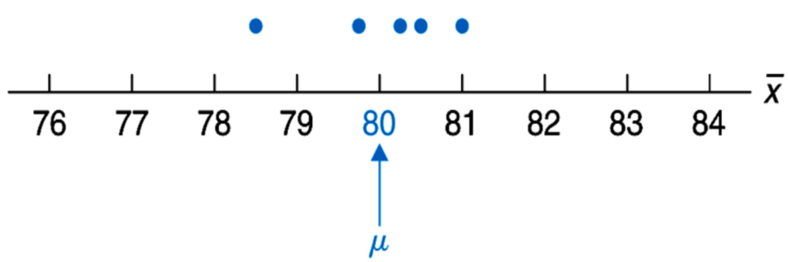
Dotplot for the sampling distribution of the mean for samples of size two (n = 2)



Possible samples and sample means for samples of size four

Sample	Heights	$\bar{x}$
A, B, C, D	76, 78, 79, 81	78.50
A, B, C, E	76, 78, 79, 86	79.75
A, B, D, E	76, 78, 81, 86	80.25
A, C, D, E	76, 79, 81, 86	80.50
B, C, D, E	78, 79, 81, 86	81.00

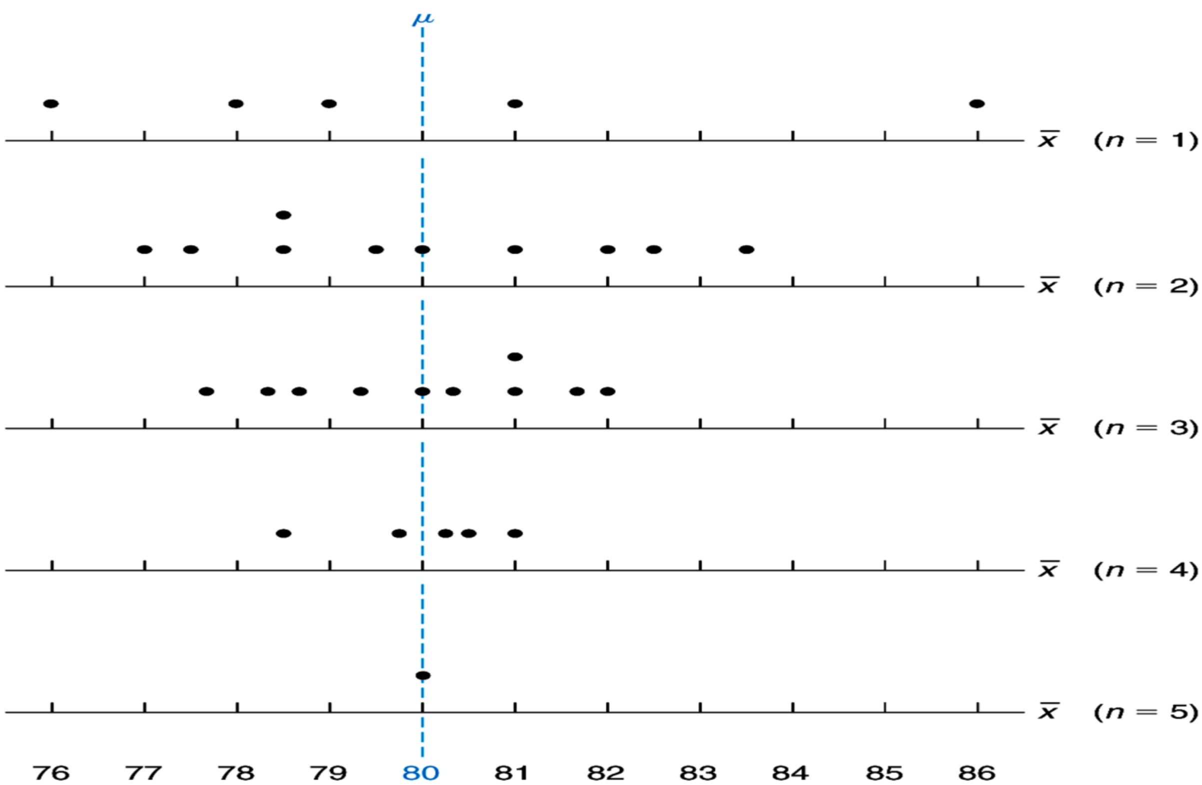
Dotplot for the sampling distribution of the mean for samples of size four (n = 4)



Sample size and sampling error illustrations for the heights of the basketball players

Sample size n	No. possible samples	No. within 1″ of μ	% within 1″ of μ	No. within 0.5″ of μ	% within 0.5″ of μ
1	5	2	40%	0	0%
2	10	3	30%	2	20%
3	10	5	50%	2	20%
4	5	4	80%	3	60%
5	1	1	100%	1	100%

Dotplots for the sampling distributions of the mean for samples of sizes one, two, three, four, and five



The Central Limit Theorem

As the sample size n increases without limit, the shape of the distribution of the sample means taken with replacement from a population with mean μ and standard deviation σ will approach a normal distribution. As previously shown, this distribution will have a mean μ and a standard deviation $\sigma/\sqrt{n}$.

For a <u>large enough</u> sample size, we can assume that $\mu_{\bar{x}} = \mu$ and $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$.

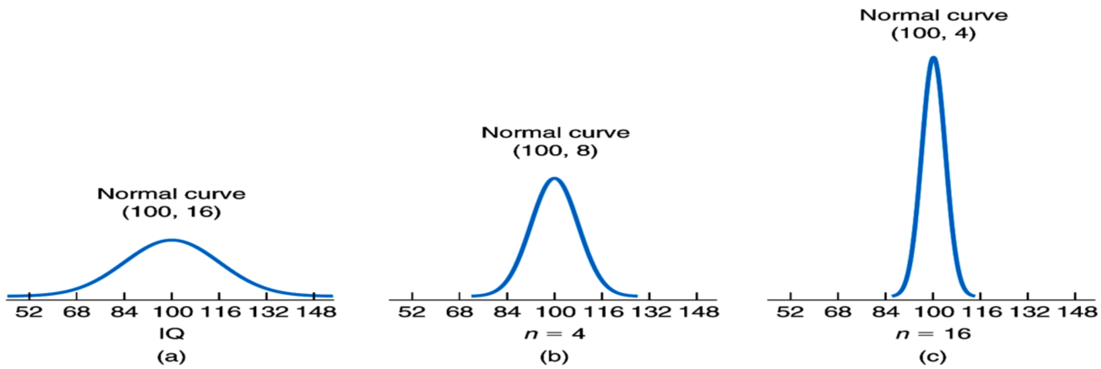
$\sigma_{\bar{x}}$ is referred to as the **<u>sampling error of the mean</u>**.

If a random sampling of size n is taken from a normally distributed population with μ and σ, then the random variable $\bar{x}$ is also normally distributed with $\mu_{\bar{x}} = \mu$ and $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$.

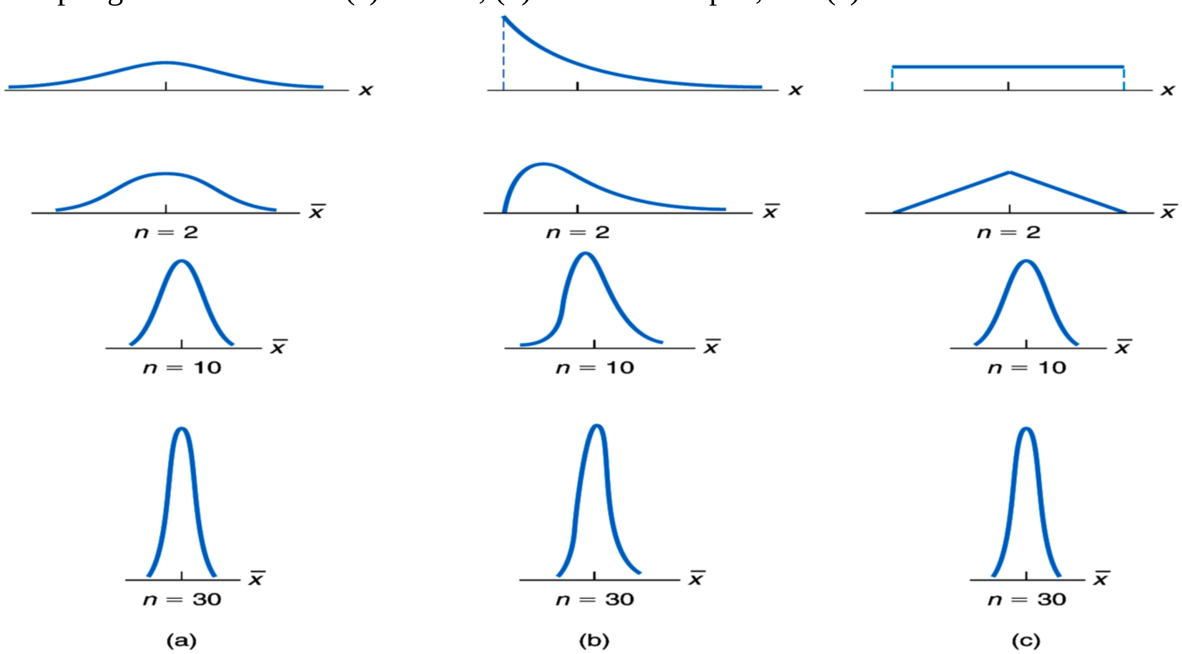
For n sufficiently large, the random variable $\bar{x}$ is normally distributed ***regardless of the distribution of the population***. The approximation is better with increasing sample size.

$n \geq 30$ **is considered to be a "large sample"**

(a) Normal distribution for IQs
(b) Sampling distribution of the mean for n = 4
(c) Sampling distribution of the mean for n = 16

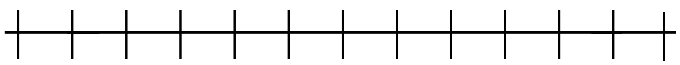


Sampling distributions for (a) normal, (b) reverse-J-shaped, and (c) uniform variables



example: The length of the western rattlesnake is normally distributed with μ = 42 inches and σ = 2.04 inches.

a) Sketch a normal curve for this population.



b) Determine the sampling distribution of the mean for random samples of size four. Draw the normal curve for $\bar{x}$ on top of the curve above.

example: Referring to the previous example, suppose a random sample of n=16 snakes is to be taken.

a) Determine the probability that the mean length $\bar{x}$, of the snakes obtained will be within 1 inch of the population mean of 42 inches, that is, between 41 and 43 inches.

b) Interpret your result in part (a) in terms of sampling error.

c) For samples of size 16, what percentage of the possible samples have means that lie within 1 inch of the population mean of 42 inches?

d) Repeat part (a) for a sample of size 50.

example: An air-conditioning contractor is preparing to offer service contracts on the brand of compressor used in all of the units her company installs. Before she can work out the details, she must estimate how long those compressors last on the average. The contractor anticipated this need and has kept detailed records on the lifetimes of a random sample of 250 compressors. She plans to use the sample mean lifetime, $\bar{x}$, of those 250 compressors as her estimate for the population mean lifetime μ, of all such compressors. If the lifetimes of this brand of compressor have a standard deviation of 40 months, what is the probability that the contractor's estimate will be within 5 months of the true mean of 62 months?

7.6 <u>Assessing Normality</u>

Method #1- Normal Probability Plots

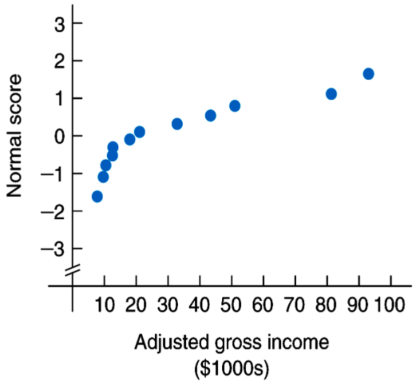
In this section we plot the sample data versus normal scores based on sample size. The idea is that we wish to know **<u>if</u>** the data is approximately normally distributed.

- If the graph is roughly linear, then accept as reasonable that the population **<u>is</u>** approximately normally distributed.
- If the graph has curves, then conclude that the population **<u>is not</u>** approximately normally distributed.

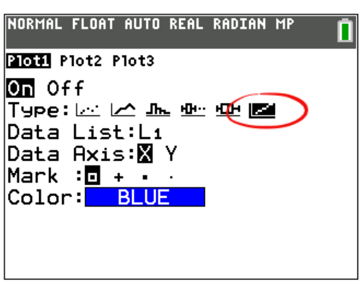
Adjusted gross incomes ($1000s)

9.7	93.1	33.0	21.2
81.4	51.1	43.5	10.6
12.8	7.8	18.1	12.7

<u>Normal probability plot</u> (also known as a normal quantile plot)for the sample of adjusted gross incomes



TI-84 Plus Directions:



Go into the StatPlot menu (2^{nd} followed by the **y=** button in the top left corner of the calculator).

Choose the **<u>last</u>** type of plot from the list.

example: In Jan. 1984, the US Dept. of Agriculture reported that a typical US family of four with an intermediate budget spent about $117 per week for food. A consumer researcher in Kansas suspected the median weekly cost was less in her state. She took a sample of 10 Kansas families of four, each with an intermediate budget, and obtained the following weekly food costs (in dollars):

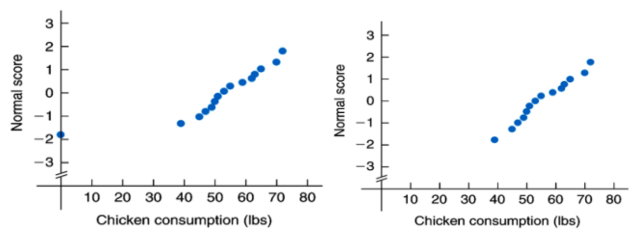
103	129	109	95	121
98	112	110	101	119

Construct a normal probability plot for the data and analyze your results.

Sometimes a normal probability plot can help you identify an outlier in a data set:

Normal probability plots for chicken consumption:

(a) original data (b) data with outlier removed



8.1 Estimating a Population Mean

A point estimate for a parameter is the value of the statistic used to estimate the parameter. For example, if we wanted to know the mean purchase price of Victor Valley homes, we might take a sample of perhaps 500 homes and compute $\bar{x}$. This would be a **point estimate** for μ, the actual mean value.

A **confidence interval estimate** of a parameter consists of an interval of numbers obtained from the point estimate together with a percentage that specifies ***how confident we are that the parameter lies in the interval***.
The confidence percentage is called the **confidence level**. If we were to draw many samples and use each one to construct a confidence interval, then in the long run, the percentage of confidence intervals that cover the true value would be equal to the confidence level.

Example: An educational psychologist at a large university wants to estimate the mean IQ of the students in attendance. A random sample of 30 students yields the following data on IQs.

107	134	101	131	108
99	132	128	106	103
101	103	113	119	111
93	109	106	102	119
99	104	126	98	112
103	103	103	116	105

a) Use the data to obtain a point estimate for the mean IQ, μ, of all students attending the university. (*Note*: The sum of the data is 3294.)

b) Is it likely that your estimate in part (a) is exactly equal to μ? Explain.

Example: Referring to the previous example, assume that the standard deviation of IQs for all students attending the university is 12.

a) Use the data from the previous example to find a 95.44% confidence interval for the mean IQ, μ, of all students attending the university.

$$\mu \pm z\left(\frac{\sigma}{\sqrt{n}}\right)$$

b) Interpret your answer to part (a) in two ways:

To find a confidence interval on the TI-83, use the **Zinterval** command in the STAT TEST menu. Enter in the appropriate information.

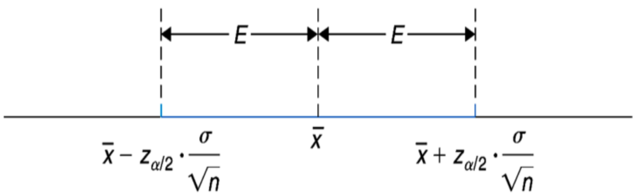
Confidence and Significance Levels

The words **confidence** and **significance** are complements of each other. When a problem has a 90% confidence level, we can also say that it has a 10% significance level. Likewise, a 95% confidence level is associated with a 5% significance level.

Sample Size

We define **E**, the **maximum error of the estimate**, to be E= $Z_{\alpha/2}\left(\frac{s}{\sqrt{n}}\right)$

E is equal to **half of the length of the confidence interval**. You might consider this to be the "plus or minus" amount usually accompanying a survey to refer to its margin of error.



- In order to get a 95% confidence level, sometimes the maximum error E must be larger than we would want. To increase the precision of our estimate, we must increase n, the sample size.

Q// How large of a sample do we take?

A// The sample size required for a particular confidence level to obtain a maximum error of the estimate E is given by the formula:

$$n = \left[\frac{Z_{\alpha/2} \cdot \sigma}{E}\right]^2$$

Example: Referring back to the previous example, you were asked to determine a 95.44% confidence interval, based on a sample of size 30, for the mean IQ, μ, of college students. Use the data from part c of the problem, after any outliers were removed.

a) Determine the margin of error E.

b) Explain the meaning of E in this context as far as the accuracy of the estimate is concerned.

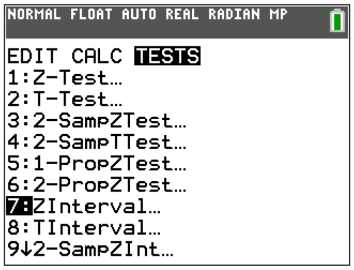
c) Determine the sample size required to ensure that we can be 95% confident that our estimate $\bar{x}$ is within 2 IQ points of μ. (Recall that $\sigma = 12$ points.)

$$n = \left[\frac{Z_{\alpha/2} \cdot \sigma}{E}\right]^2$$

d) Find a 95% confidence interval for μ if a sample of the size determined in part (c) yields a mean of $\bar{x} = 112$.

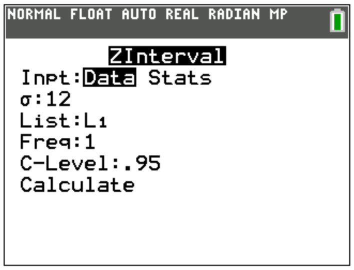
Why was the mean value of 109.8 IQ points changed to 112 IQ points in order to answer part d?

Using your TI-84 Plus calculator:

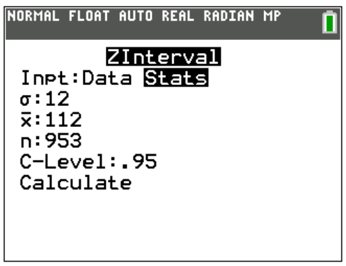


Press **Stat** and then highlight the **Tests** menu.

Select **7: ZInterval**.



If your data is in a list, select **Data** in the top row and make sure the list containing your data is typed in.

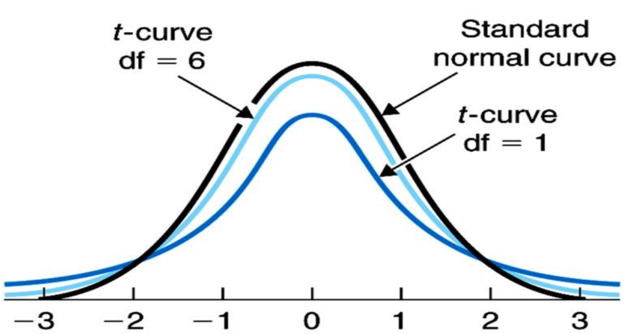


If you don't have the raw data, select **Stats** in the top row and enter the requested values.

8.2 <u>t-Curves</u>

When a large sample is impractical, impossible, or too costly, a t-curve is used. We say that the t-curve has ***n-1 degrees of freedom*** $(df = n-1)$.

The t-curve is a very <u>robust</u> measure: it is very sensitive to departures from the assumptions. This is because there is a different t-curve for each sample size.



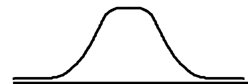
Standard normal curve and two *t*-curves

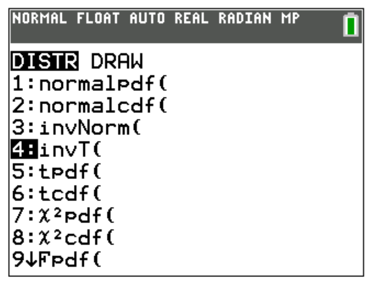
For t-curves we <u>must</u> assume that the sample is taken from a population that is already normally distributed. To check this assumption, you must sometimes create a normal probability plot or a modified boxplot.

<u>Properties</u>

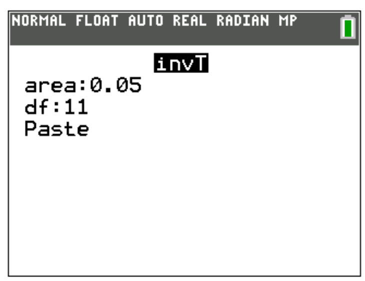
1. The total area under the t-curve is equal to 1.
2. A t-curve extends infinitely along the x-axis to both the left and right.
3. A t-curve is symmetric about t=0.
4. As the number of degrees of freedom increases, t-curves look increasingly like the standard normal curve.

- t_α represents the area to the **<u>right</u>** of t under the t-curve:

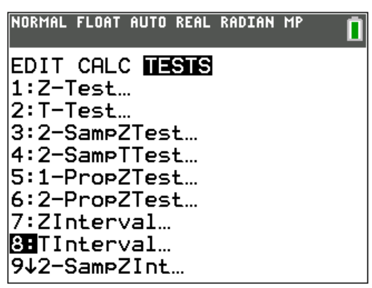




To find a t-value, select the **dist** button (above **vars** key) and use the **4: invT** choice.



Make sure the area you enter is to the right of the *t*-value.



For confidence intervals, press **Stat** and then highlight the **Tests** menu.

Select **8: TInterval**.

Example: The mean annual subscription rate for law periodicals was $29.66 in 1983. A random sample of 12 law periodicals yields the following annual subscription rates, to the nearest dollar, for this year.

30	46	44	47
42	38	62	55
52	48	43	54

a) Determine a 95% confidence interval for this year's mean annual subscription rate μ for all law periodicals. (Note: are $\bar{x}$ =46.75 and **s**=8.44.)

b) Does your result from part (a) suggest an increase in the mean annual subscription rate over that in 1983?

Which should I use… a t-curve or a z-curve?

Example: A meteorologist who sampled 13 thunderstorms found that the average speed at which they traveled across a certain state was 15 miles per hour. The standard deviation of the sample was 1.7 miles per hour.

a. Find a 99% confidence interval for the mean.

b. If a meteorologist wanted to use the highest speed to predict the times it would take storms to travel across the state to issue warnings, what speed would she likely use?

Recall… whenever we use a t-curve, we must assume the data is approximately normally distributed. If we are unsure, we must create a normal probability plot or a modified boxplot to check this assumption.

8.3 <u>Population Proportions</u>

Suppose we wish to know what proportion of a population has a particular attribute.

Let **p** = population proportion

$\hat{p}$ = sample proportion

Formula: $\hat{p} = \frac{x}{n}$

Note that **p** is a statistic being used to make a prediction about the population parameter $\hat{p}$.

example: If 108 families were sampled to see if they have a microwave and x=102 responded "yes," then $\hat{p} = \frac{102}{108}$.

Suppose a large random sample of size n is to be taken from a 2-category population with population proportion **p**. Then the random variable $\hat{p}$ is approximately normally distributed with $\mu_{\hat{p}} = \mu$ and $\sigma_{\hat{p}} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

<u>Assumptions:</u>

- a simple random sample was taken
- the population is at least 20 times larger than the sample
- the items in the population are divided into two categories
- the samples must contain at least 10 individuals in each category

The **<u>margin of error E</u>** for the estimate of **p** is given by **E**= $Z_{\alpha/2} \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$. (It is equal to half of the confidence interval.)

example: Studies are performed to determine the percentage of the nation's 10 million asthmatics who are allergic to sulfites. In a recent survey, 38 of 500 randomly selected U.S. asthmatics were found to be allergic to sulfites.

a) Determine a 95% confidence interval for the proportion, p, of all U.S. asthmatics who are allergic to sulfites.

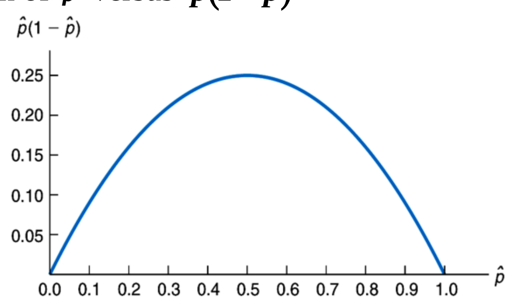
b) Interpret your results from part (a).

<u>Sample Size:</u>

To determine the proper sample size to match the margin of error with the confidence level, first determine whether $\hat{p}$ (or an estimate for $\hat{p}$ **)** is known or not. Use **n**= $\hat{p}(1-\hat{p})\left(\frac{Z_{\alpha/2}}{E}\right)^2$, then round up to the nearest integer when $\hat{p}$ is <u>known</u>.

Use **n**= $0.25\left(\frac{Z_{\alpha/2}}{E}\right)^2$, rounded up to the nearest integer when a guess for $\hat{p}$ is <u>unknown</u>.

Graph of $\hat{p}$ versus $\hat{p}(1-\hat{p})$



example: Referring to the previous example (U.S. asthmatics),

a) Determine the margin of error for the estimate of p.

b) Obtain a sample size that will ensure a margin of error of at most 0.01 for a 95% confidence interval without making a guess for the observed value of $\hat{p}$.

c) Find a 95% confidence interval for p if for a sample of the size determined in part (b), the proportion of asthmatics allergic to sulfites is 0.071.

d) Determine the margin of error for the estimate in part (c) and compare it to the margin of error specified in part (b).

Example: 35% of adults in America eat pizza at least once per week. A random sample of 250 adults in a medium-size college town were surveyed, and it was found that 110 eat pizza at least once per week. Estimate the true proportion of American adults who eat pizza at least once per week with 90% confidence and comment on your results.

A recent study indicated that 29% of the women of the 100 women over age 55 in a study were widows.

a. How large a sample must you take to be 90% confident that the estimate is within 0.05 of the true proportion of women over age 55 who are widows?

b. If no estimate of the sample proportion is available, how large should the sample be?

